HOW TO DIVIDE EXACTLY IN MATHEMATICS

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Abstract. In this work, motivated by the fact that the division $1/3 = 0,\widehat{3}$ is not exact, while for example one can divide a cake into three exact pieces. We describe a logical procedure to divide every real number $r \in \mathbb{R}$ into $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ or more general $m \in \mathbb{R} \setminus \{0\}$ exact parts and we also show the corresponding exact formula.

1. Introduction.

Arithmetic is one of the fundamental tools in Mathematics and also in most of the sciences.

Through the history, different civilizations have developed on their own the necessary knowledge to solve their arithmetic questions. As far as arithmetic is concerned, the main civilizations that have contributed to its expansion and diffusion have been the Babylonians, Egyptians, Greeks, Hindus and Arabs. Each one has distinguished itself by its particular contribution. Among them, the Hindu culture stands out in an important way, since they are due to the current positional decimal numbering, later transmitted by the Arabs. These people proposed their respective methods for solving the four elementary operations of arithmetic.

One of the most important is the division, see [2] and its references for a long explanation and a comparison between their methods through the history, among other topics. The purpose of this manuscript is not to enter into the history of the division, but it is to enter into a concept that actual mathematics gives not an answer about exact division, as a consequence the main goal is to provide a logical procedure (or visual proof) to divide exactly and give the corresponding formula that in general mathematicians do not answer to that question and talks about the imprecision of Mathematics.

We start by describing the procedure for the simplest case $1/3$ and next we extend it to the general case.

The story of this work starts when a non-mathematician friend, A. Ratia, knowing that I am mathematician asked me the following question: How one can divide 1 into three equal parts with the result an exact number in Mathematics? By exact number we understand that the division has zero-rest or equivalent, the division is exact. Its idea was that since one can divide a cake into three equal exact pieces, why mathematics says that $1/3 = 0,\widehat{3}$ which clearly is not an exact number although the division is well done by mathematics since

$$3 \cdot 1/3 = 3 \cdot 0,\widehat{3} = 0,\widehat{9} = \sum_{k=0}^{\infty} \frac{9}{10^k} = 9\frac{1/10}{1-1/10} = \frac{9/10}{9/10} = 1.$$

I asked some colleagues and they gave vaguely ideas saying that mathematics are not exact and many other things that are not of interest to reproduce here, but none gave me the answer. I

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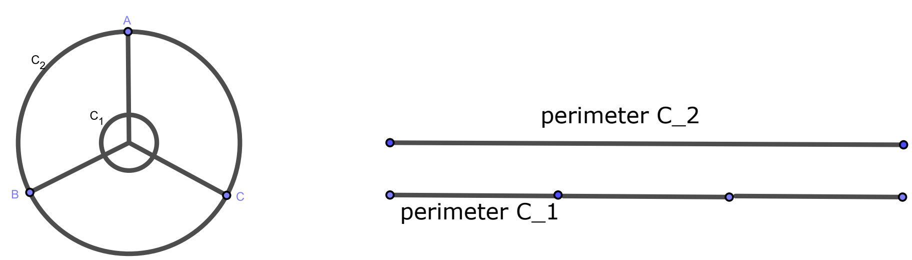
want to mention that this result is new and I have not talked to one mathematician that knew the answer. After some time I had an accident and some time to think about that question, I solved, and here is the final and exact answer. The answer is presented in the same manner as I thought and solved.

First I am going to explain a simple logical argument to see that it is possible to divide the cake into three exact equal pieces.

Theorem 1. *Assume that we have a cake represented by a circle. We can divide the cake into three equal pieces.*

Proof. Consider a circle C_1 of perimeter ℓ (see the following figure). Now consider three equal intervals of length ℓ. We denote the circumference C_2 with same center as C_1 (the origin without loss of generality) of perimeter 3ℓ which has fixed each part of length ℓ, now we join each mark of length ℓ in the perimeter of the circumference C_2 with the same center and finally, C_2 as well as C_1 are divided into three exact equal pie

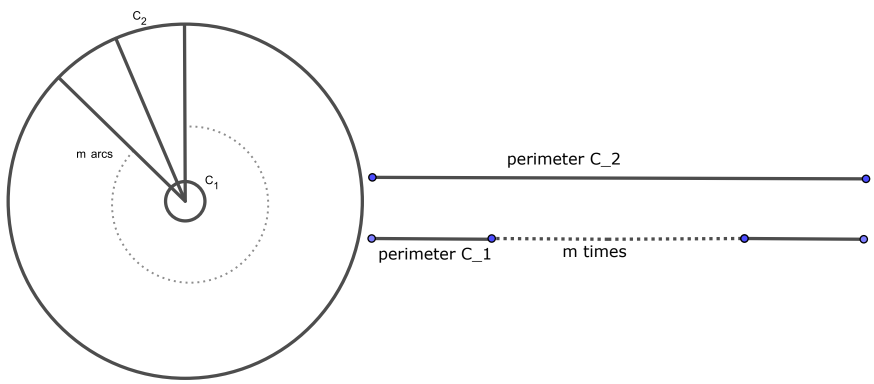


- The exact formula. Fortunately the decimal base is not the unique one in Mathematics. As a consequence, using the classical notation $r_{(s)}$ as r expressed in s-base where $r \in \mathbb{R}$ and the base $s \in \mathbb{R} \setminus \{0\}$, (cf. [1]). We write $\frac{1}{3}_{(10)}$, i.e., $1/3$ in decimal base is equal to $0,1_{(3)}$, i.e., $0,1$ in 3-base, which is exact since the division in 3-base is exact.

The general procedure is obvious, to do so, we consider as before r/m with $r \in \mathbb{R}$, and a natural number $m \in \mathbb{N}$:

Theorem 2. *Assume that we have a cake represented by a circle. We can divide the cake into m equal pieces.*

Proof. Assume that we have a cake represented by a circle with a circumference C_1 of perimeter ℓ (see the previous figure). Now consider m equal intervals of the same length ℓ. We denote the circumference C_2 with same center as C_1 (the origin without



loss of generality) of perimeter $m\ell$ which has fixed each part of length ℓ, now we join each mark of length ℓ in the perimeter of the circumference C_2 with the same center and finally, C_2 as well as C_1 are divided into m exact pieces.■

- The exact formula. As before, using the classical notation $r_{(s)}$ as above, $\frac{r}{m}_{(10)} = 0,r_{(m)}$, i.e., $0,r$ in m-base. That is the exact division is exact in the m-base.

Remark 3. *Assuming that $m \in \mathbb{R} \setminus \{0\}$, one can prove the same results without any relevant change.*

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